Maximal Evenness as Conceptual Apparatus for a Course on Post-Tonal Theory and Analysis

Adam Ricci

In "Music Theory's New Pedagogability," Richard Cohn observes that as recently as 25 years ago "the boundary between research and teaching at the introductory levels seemed inevitable and unbreachable…[and that]…we are at a somewhat different juncture now. A number of central concepts have emerged…that can be taught at the introductory level" (Cohn 1998, [3]). Among the central concepts that he mentions is Clough and Douthett's concept of maximal evenness. A maximally even set "is a set whose elements are distributed as evenly as possible around the chromatic circle" (Clough and Douthett 1991, 96). Each maximally even set is labeled ME(c,d), where c is the cardinality of the universe—in the case of the chromatic scale, 12—and d is the cardinality of the set contained within it. Timothy Johnson's textbook *Foundations of Diatonic Theory*, as if in answer to Cohn's call, puts maximal evenness front and center; indeed, Johnson himself acknowledges the connection in his Preface. As to the intended audience, the "material in this text was originally designed for use as a supplement in traditional Theory I courses, but…is equally appropriate for courses in the fundamentals of music…and for stand-alone courses [in]…mathematics and music" (Johnson 2003, vii). Maximal evenness can also profitably be used as a central concept for a course on twentieth-century music theory and analysis. In such a course, typically the final one in the undergraduate curriculum, students possess substantial music theory knowledge, and therefore maximal evenness can serve synthetic ends, as a means of consolidating (and expanding) what students already know. Such synthetic ends are appropriate in a course that often constitutes the conclusion of a student's formal training in music theory.

This paper will outline such a course, based on one that I teach at my own institution. In this course, instead of introducing theoretical concepts using musical examples, a typical strategy in required music theory courses, the focus is more on music theory—on exploring the underlying structure of the objects students have been studying and tasks students have been carrying out for the previous two years. What I offer here is a new way to organize the final course in the undergraduate theory curriculum, a way that suggests a different ordering and combination of topics—and manner of talking about topics—than that offered by current approaches. Central to the course is the concept of maximal evenness, but woven in are closely related ideas from neo-Riemannian theory and elementary combinatorial theory.

Using maximal evenness as a central concept has several benefits. First, it serves as a core idea about which students can integrate knowledge. The same sets are encountered in different contexts, each of which reveals a bit more about them.[1] Second, it fulfills a necessary component of what Ken Bain (2004, 100) terms a "natural critical learning environment": namely, "an intriguing question or problem" that students seek to answer/solve beginning with the first day of class. The intriguing problem is creating maximally even sets, then considering the ways in which they might be musically relevant. Third, since there is one and only one maximally even set-class of a particular cardinality, the family of maximally even sets is small.[2] Thus, these sets can serve as prototypes to which other sets may be related.[3] Fourth, maximal evenness can smooth the transition between familiar tonal objects—triads and 7th chords—and less familiar sets that are suited to both tonal and post-tonal environments.[4] Maximally even sets include both chords and scales, the difference between the two having to do with lesser or greater cardinality relative to the universe.

Maximally even sets can be expressed as pitch classes or durations, both of which will be explored in this paper. Part I. will outline the initial inquiry into maximal evenness, which proceeds by cardinality within the 12-pitch-class universe. Though Johnson proceeds similarly, the differences in my approach—including

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[1] For a twentieth-century course, Rogers (2004, 73) argues "that some approach be chosen—that a rationale exists—so that a course…is bound together by a…governing philosophy that provides some bearings—and that students perceive what that philosophy is." For a different central concept, see Sallmen 2001.

[2] Maximally even sets can be conceived in a universe of any size, so the size of the family depends upon the size of the universe; in this paper, of course, the focus will be upon the 12-pitch-class universe, but I will also discuss some smaller universes that are musically relevant.

[3] Quinn 2006-07 argues for the prototypical status of ME sets.

[4] Mancini 1991 and Pople 2004 suggest some other ways to link set-theoretic concepts with tonal materials.

using students as pitch classes,[5] incorporating mode, and paving the way for pitch-class set theory—merit a brief synopsis. Because I will be discussing my interactions with students, the tone of Part I. will be somewhat informal. Part II. first explores maximal evenness in the 7-letter-name universe; second, through a comparison of sets in the diatonic and chromatic universes, illustrates a new method for teaching enharmonic modulation that has points of contact with recent neo-Riemannian theory; and third, studies maximal evenness in the rhythmic domain. Part III. discusses how, with the aid of elementary combinatorics, maximal evenness can be integrated with other set-theory topics typically covered in the twentieth-century theory course, including transpositional combination and transpositional and inversional symmetry.

I. The Initial Inquiry

The course begins with a determination of the maximally even (henceforth, ME) sets in the chromatic scale, proceeding by cardinality. I tell students we are going to conduct an experiment, and ask them to place 12 chairs in a circle.[6] As they arrange the chairs, I ask what significance the number 12 might have for our musical studies, and several students typically answer that there are 12 pitches in the chromatic scale. Next one student volunteer chooses a chair to sit on. I hand this individual a poster board with the letter "C" on it to hold up. This initial set illustrates the concept of a *trivial* ME set: there is only one way to create a one-element set, and it is *de facto* maximally even (Clough and Douthett 1991, 145).[7] I then ask a second student volunteer to take a seat as far away from the first individual and a third volunteer to choose the appropriate poster board from the pile to hand to this person (F♯/G♭). This second set is non-trivial because the second individual's position is strictly determined by where the first individual sits. Continuing

[5] Johnson instructs students to take pencil to paper, although he does use the dinner-table analogy at one point (pp. 13-14).

[6] At my own institution, I am fortunate to have a ready-made space, a large outdoor rotunda that contains 12 benches, each bench affixed in front of one of the 12 structural pillars supporting the organ hall above. Because the students are in the interior of the circle, they must continually turn around to visualize the sets, keeping them actively involved.

[7] Trivial ME sets include those of cardinality 0, 1, c–1 and c, where c is the size of the universe.

chronologically by cardinality, I ask the third student to take a seat such that each student has an equal amount of personal space. At this point, it becomes apparent that one or both seated students needs to move. Choosing the more efficient way, I say we should keep "C" in place. Once the other two students are correctly seated, another student hands the appropriate poster boards (E and G♯/ A♭) to them as F♯/G♭ is returned to the stockpile.

As we proceed by cardinality, I compile information for each set on a whiteboard, information that is later provided to students as a handout given as Example 1. Example 1a contains explanatory text, providing a "key" to the information in Examples 1b and 1c.

A maximally even set "is a set whose elements are distributed as evenly as possible around the chromatic circle" (Clough & Douthett 1991, 96).

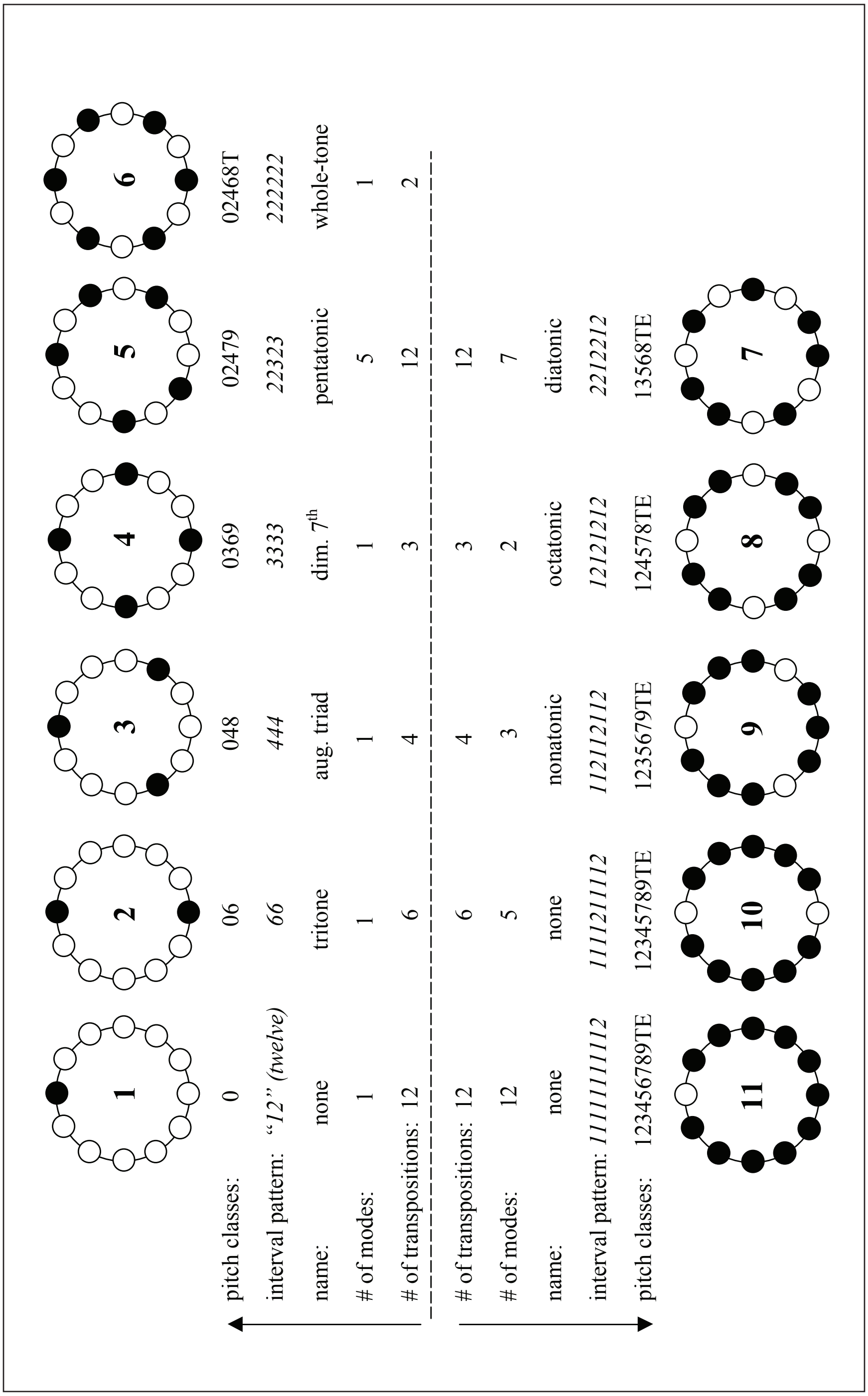
In Example 1b are all the maximally even (ME) sets in the chromatic scale, with the exception of the null set and the aggregate.

The cardinality of each set—the number of objects it contains—is indicated in the center of the circle. The filled circles indicate pitch classes in the set, and the unfilled circles indicate pitch classes omitted from the set.

Complementary ME sets are aligned vertically. ME(12,6)—the six-element ME set in the chromatic scale—is its own complement. Each remaining set is drawn so that filled and unfilled circles exchange places in complementary sets.

- pcs = pitch classes (the 12-o'clock position is "0")
- interval pattern = an ordered list of the distance between successive members of a set (0 = unison, 1 = half step, etc.)
- name = familiar name
- # of modes = the number of distinct rotations of *the interval pattern*
- The number of modes is equal to $\frac{s}{\gcd(c,s)}$, where
  - c = the number of chairs in the circle and
  - s = the number of students
  - gcd(c,s) = the greatest common divisor of c and s
- # of transpositions = the number of distinct rotations of *the seating position*
  - The number of transpositions of an ME set is the same as the number of transpositions of its complementary set
  - The number of transpositions is equal to $\frac{c}{\gcd(c,s)}$

Example 1a: Handout on maximal evenness: Explanatory text



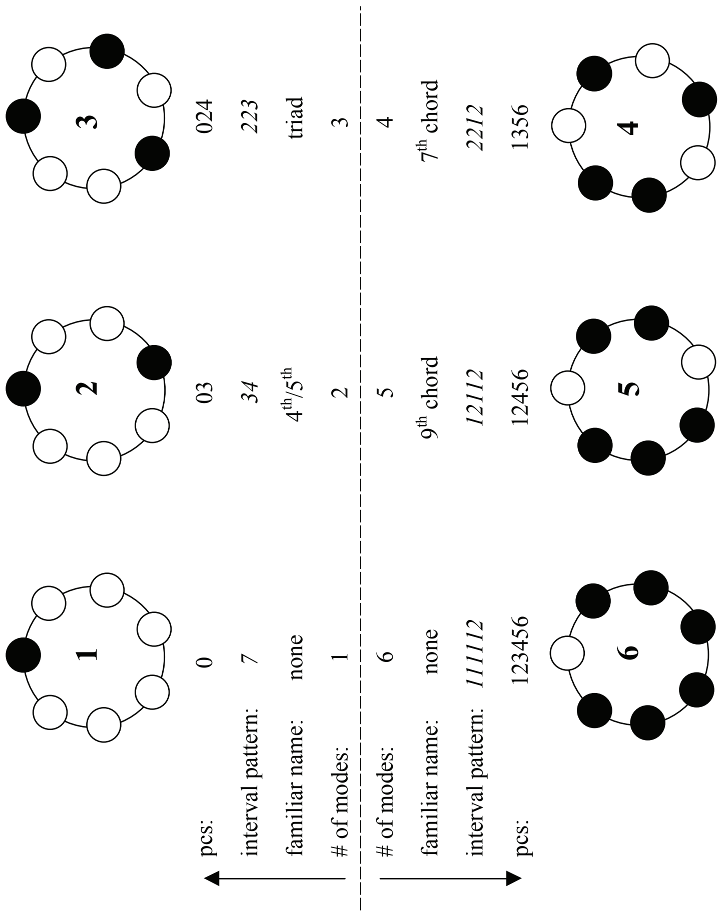
	1	2	3	4	5	6
pitch classes:	0	06	048	0369	02479	02468T
interval pattern:	*"12" (twelve)*	*66*	*444*	*3333*	*22323*	*222222*
name:	none	tritone	aug. triad	dim. 7th	pentatonic	whole-tone
# of modes:	1	1	1	1	5	1
# of transpositions:	12	6	4	3	12	2

	11	10	9	8	7
# of transpositions:	12	6	4	3	12
# of modes:	12	5	3	2	7
name:	none	none	nonatonic	octatonic	diatonic
interval pattern:	*11111111112*	*1111211112*	*112112112*	*12121212*	*2212212*
pitch classes:	123456789TE	12345789TE	1235679TE	124578TE	13568TE

Example 1b: Handout on maximal evenness: ME(12,n)

The number of transpositions of a ME(7,*s*) set is always 7, since the greatest common divisor of 7 and any other integer is 1. For this reason, this information is left out.

The number of modes of a ME(7,*s*) set is always *s* (i.e., the number of modes is equal to the cardinality of the set), for the same reason!



Example 1c: Handout on maximal evenness: ME(7,n)

In Example 1c are all the maximally even sets in the diatonic scale (again, with the exception of the null set and aggregate). All integers should be thought of in terms of seven total pcs, so, e.g., 024 = CEG.

First we list the pitch-class/name-class designations[8] given on the poster boards ("C", "C F♯/G♭", "C E G♯/A♭", etc.); second, the interval pattern for each set, measured by a student who traverses the spaces between seated students. In undertaking this measurement, the question naturally arises as to whether to count both, only one, or neither occupied chair in a pair. I suggest that it does not matter which way we count provided we are consistent. I remind the students that in tonal theory, we count both endpoints—what I call the "eight days a week" counting method (i.e., Monday-to-Monday equals an octave)—and tell them that in twentieth-century theory, we count only one endpoint, which makes interval addition easier. Thus, each interval is measured by starting behind an occupied chair and counting chair-to-chair "steps." The corresponding interval patterns for the first three sets are "12," "66," and "444."[9] Third, I ask students to determine both how many different rotations of the seating pattern there are and how many different vantage points there are from which to view the set—assuming one always stands behind an occupied chair. In order to count the different rotations of the seating pattern, the seated students all shift clockwise by one seat repeatedly until the same set of benches is occupied. In order to count the unique vantage points, one student walks around the circle reading off the interval pattern clockwise from each vantage point until they return to their starting point. I act merely as scribe, writing these patterns on the whiteboard; the number of unique interval patterns can then be counted by pruning duplicates.

I also ask what the familiar name of these two procedures are—"transposition" usually comes fairly easily; "mode" usually does not. Rather than supply the term right away, I continue to say "different vantage points" or "distinct rotations of the interval

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[8] Brinkman 1986 defines "name class" and outlines a representation of letter-name-plus-accidental labels in terms of a pair of integers, one representing the pitch class and one representing the letter-name class. Harrison 2002 calls such labels "notational tokens" for a particular pitch class.

[9] Rather than inserting commas between each interval, I clarify that "12" means "twelve" and not "one, two," the only case in which this potential ambiguity arises.

pattern" until the second class day to allow the students to furnish the term on their own and to teach them the benefit of terminology. At some point in the exploration, after some arguments about the spelling of particular pitch classes arise, I suggest that it might be handy to use a labeling system that does not require us to choose particular spellings, at which point I ask the seated students to turn over their poster boards, revealing integer labels. I continue to represent sets with letter names and accidentals as well, however, for the sake of familiarity (for reasons of space, these labels are omitted from the handout). Finally, I ask if any students recognize each set—whether it has a familiar name.

Like ME(12,2) and ME(12,3), ME(12,4) equally divides the chromatic; such sets are Clough and Douthett's Class A ME sets, whose interval patterns contain only one interval size. The three classes and their characteristics are given in Example 2.[10]

Class	Clough & Douthett (1991, 97)	Sample interval patterns	Cardinality
A	gcd(c,s) = s, s ≠ 1	xx, xxx, etc.	all possible
B	1 < gcd(c,s) < s	xyxy…, xxyxxy…, etc.	all possible
C	gcd(c,s) = 1	x	1
		xy	2
		xxy, xxyxy, xxxyxxyxxy	3 through c–1

Example 2: The three classes of ME sets

Class A ME sets contain two or more of a single interval size; Class B and C ME sets contain two different interval sizes that differ by one chromatic step. Whereas the interval pattern of any Class B ME set can be decomposed into a repeating pattern (e.g. *xy* times 2, *xyy* times 2), that of a Class C ME set cannot.

Because 5 does not equally divide 12, determining ME(12,5) is more difficult. Often multiple solutions are proposed, and I write all of them down. Upon examining the interval patterns, students

[10] Although I do not in the class invoke the three classes of ME sets outlined by Clough and Douthett, I will find it useful to do so in this paper. It is of course a singular advantage of the concept of maximal evenness that it includes both equal and unequal divisions of an equal-tempered space. Quinn (2006-07, 5-9) proposes a division of Class B into two subtypes: one in which gcd(c,s) = c–s ≠ 1, and another in which gcd(c,s) ≠ c–s ≠ 1. The latter subtype does not arise in the 12- or 7-element universes, the ones I focus upon here.

quickly realize that the multiple solutions are equivalent to within transposition. We observe that the interval pattern consists of two different intervals alternating irregularly (23223 or some rotation thereof), and I point out that the "3"s are as equally spaced among the "2"s as possible—and that this is a useful way to check that a set (whose interval pattern contains two different intervals) is indeed maximally even.[11] ME(12,5) constitutes a Class C ME set. ME(12,6) is Class A.

For sets of larger cardinality, I mention that there is a shortcut, providing an opportunity to introduce the idea of complementation. For this reason, complementary ME sets are aligned with each other in the handout; ME(12,6) is its own complement. Strictly speaking, the clockfaces in one half of the handout are not even necessary, because one can simply imagine the filled circles to be unfilled and the unfilled ones to be filled. Thus, although I encourage students to construct these larger sets on their own, I do not spend time in class actively constructing them but instead find them by way of their complements. After compiling all of this information in the first week of class, I casually mention that there are handy formulas for calculating the number of transpositions and modes of an ME set, and that these formulas involve the number of chairs, the number of students, and the greatest common divisor. That complementary ME sets have the same number of transpositions provides a big hint for figuring out the formulas, and one or two students usually figure them out.[12] The formulas are given in Example 1a. The existence of these simple formulas for the number of transpositions and modes of an ME set helps to reinforce these concepts.[13]

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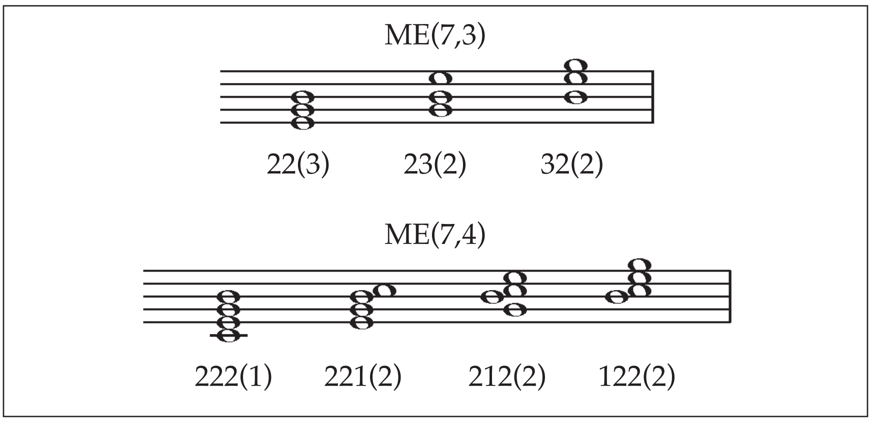
[11] Johnson has a more lengthy discussion of the problems arising in constructing ME(12,5) and ME(12,7). He foregrounds a solution in which equally spaced dots are first placed around the circle without regard to the 12 fixed positions—in other words, every 12/5 "hours"—then rounded to the nearest fixed position. In his discussion of the dinner table analogy (pp. 13-14), he suggests a solution that is equivalent to mine.

[12] The loose locution "complementary ME sets have the same number of transpositions" hints at the fact that ME sets point the way to the concept of set class.

[13] The formula for the number of distinct pcsets in ME(c,d) appears as Clough and Douthett's Theorem 1.9 (p. 113); the formula for the number of distinct modes of ME(c,d) appears as Lemma 1.15 (p. 114), which is defined as the number of pcsets in ME(c,d) that contain a particular pc. (The connection with mode is explained on p. 116.) The formula for the number of distinct pcsets in ME(c,d) appears prominently and early in Johnson 2003 (p. 6), but mode is not discussed at all.

II. Maximal Evenness in the Seven-Element Universe, Enharmonicism, and Rhythm

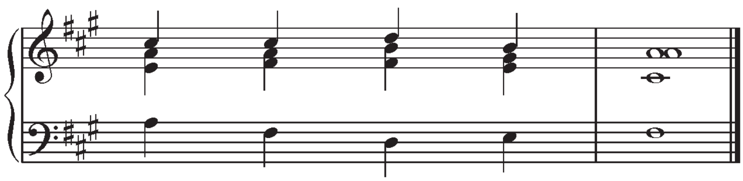
In order to illustrate the generality of the concept of maximal evenness, I explore the universe of seven elements as well; see Example 1c. Among the sets here are the triad (of indeterminate quality) and 7^{th} chord (of indeterminate quality), which are abstract complements of one another.[14] Examining maximal evenness in the seven-element universe helps students integrate knowledge in at least two ways. First, as shown in Example 3, the different inversions of close-position triads and 7^{th} chords are, from this perspective, their different modes.



Example 3: Triad and 7^{th}-chord positions as modes of ME(7,3) and ME(7,4)

Second, students' understanding of root motion can be used to teach the concept of pitch-class transposition. Example 4 demonstrates the connection using a normative tonal progression that incorporates the three most common diatonic root motions: by descending 5^{th}, descending 3^{rd}, and ascending 2^{nd}.

[14] Incidentally, the voice leading of the progression vii$^{\circ 7}$-to-I serves as a great model of these two sets as *literal* complements.



A:	I	vi	ii⁶	V	vi
Root motion:	↓3rd	↓5th	↓5th	↑2nd	
Pitch-class sets:	{A,C♯,E}	{F♯,A,C♯}	{B,D,F♯}	{E,G♯,B}	{F♯,A,C♯}
Transposition, **mod 7**:	T_5	T_3	T_3	T_1	

Example 4: Diatonic transposition

Diagramming root motions on the 7-letter-name circle reveals an inconsistency in labels for root motions. While the labels "root motion by (ascending or descending) 2nd" and "root motion by (ascending or descending) 3rd" correspond to the shorter distance between the two locations on the circle, "root motion by (ascending or descending) 5th" corresponds to the *longer* distance around the circle. That we use "5th" rather than "4th" for such root motions no doubt relates to the harmonic series—to notions of moving from the fundamental to the second partial and vice versa. This inconsistency in root-motion labels provides an opportunity to point out that in twentieth-century theory pitch-class transposition is always reckoned *clockwise*; in that respect it is more consistent and therefore easier than the conventional method of labeling root motions.

The difference between ME sets of a given cardinality in the 7-element universe and in the 12-element universe provides a context for teaching enharmonic modulation, often the final harmony topic in tonal theory. Thus, maximal evenness can be used either to introduce the topic or to review and flesh out a recently treated topic. The main observation to be made here is that though ME(12,3) and ME(12,4) have undifferentiated interval patterns (444 and 333, respectively), ME(7,3) and ME(7,4) consist of an unequal mix of different intervals (223 and 2221, respectively). In order to *spell* any instance of ME(12,3), then, one must choose one of the three modes of ME(7,3), corresponding to the three "inversions" of an augmented triad. For example, pc set {048} can be spelled in "root position" (CEG♯, 223), "first inversion" (CEA♭, 232), or "second inversion" (CF♭A♭, 322). Similarly, in order to spell any instance of ME(12,4), one must choose one of the four modes of ME(7,4), corresponding to the four inversions of a diminished-7$^{\text{th}}$ chord.

This perspective on enharmonicism helps sensitize students to the importance of spelling for understanding enharmonic modulation, providing an entrée to that topic. It also helps students distinguish between "structural" enharmonicism and notationally convenient enharmonicism.[15] In the context of enharmonic modulations using augmented triads or diminished-7$^{\text{th}}$ chords, "structural" enharmonicism involves changing the mode (i.e., "inversion") of the chord. Thus, a transformation such as {F,A♭,C♭,E𝄫}→{E♯,G♯,B,D} would therefore *not* be considered structural, because it does not involve a change of mode.[16] Enharmonic modulation using the diminished-7$^{\text{th}}$ chord is well understood and prominently featured in most textbooks, so I need not review it here. Enharmonic modulation using the augmented triad, however, is rarely treated in current textbooks,[17] but recent work in neo-Riemannian theory suggests a systematic framework for teaching it. Example 5 lists all possible parsimonious voice leadings between augmented triads and major and minor triads.[18] By parsimonious voice leading, I mean voice leadings in

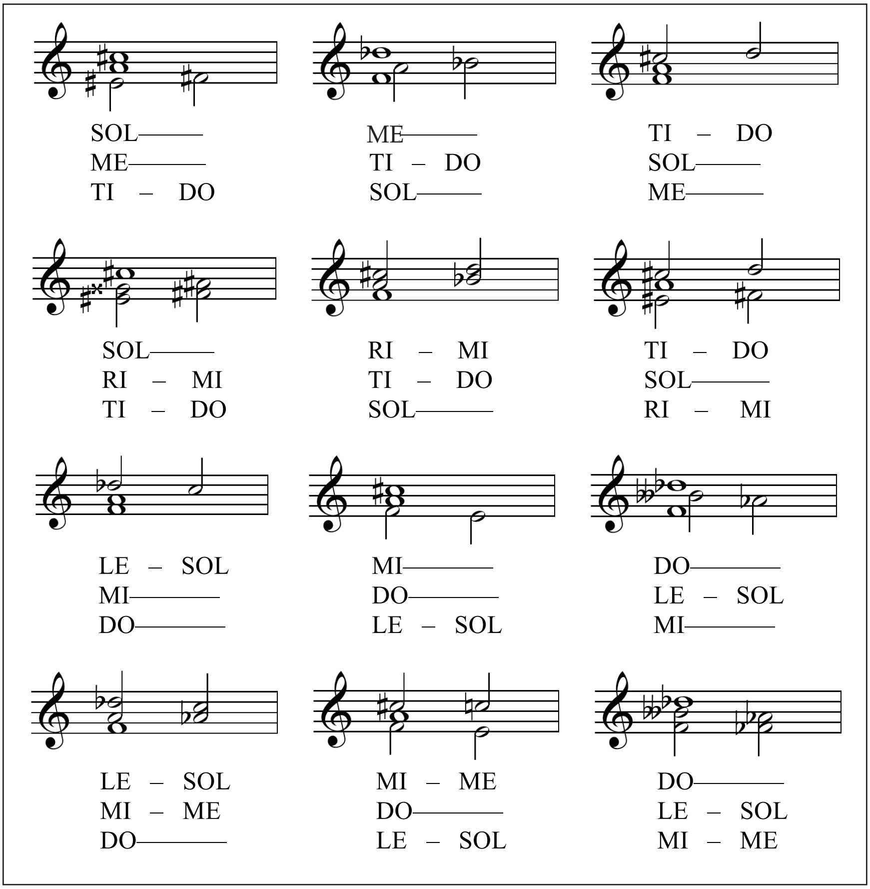
[15] See Harrison 2002 for a rich discussion of the issues involved in interpreting enharmonic passages.

[16] Some textbooks list more than four notational exemplars of a particular transposition of ME(12,4) in an attempt to be relatively comprehensive—but without clarifying the differences between the two types of enharmonicism. The example above would be structural enharmonicism only in the context of an equal division of the octave, in which enharmonic reinterpretation is necessary to return notationally to one's starting point. See Cohn 1996, 9-11. Atlas 1983 calls this "type 2" enharmonicism, in which the precise point of enharmonic reinterpretation cannot be determined. An excerpt from Gabriel Fauré's song "Toujours" that I will discuss (see Example 8) includes one example of type 2 enharmonicism.

[17] A notable exception is Roig-Francolì 2003 (830-34), which discusses how augmented triads can be reinterpreted as V^{+} of different tonics and illustrates such an enharmonic modulation in Wolf's "Das verlassene Mägdlein." This text is also the first to include the neo-Riemannian operations P, L and R (pp. 863-71).

[18] Weitzmann 1853 lists these twelve voice-leading possibilities in six different examples (pp. 10-11, 11, 18, 23, 24 and 28, respectively), but groups them in four different ways, none of which matches the organization of my Example 5. Two of his examples begin with consonant triads, while the remaining ones begin with either a single

which each moving voice shifts by only one semitone.[19]



Example 5: All parsimonious voice leadings from pcset {159} to a major or minor triad

The first row lists voice leadings in which one voice ascends, and the second row those in which two voices ascend; the third and fourth rows list the complementary voice leadings involving descending voice motion. Stationary voices are realized as unisons (whole notes), while—with one exception—moving voices are

augmented triad or all four augmented triads in succession. Various of Weitzmann's examples are reproduced (some with changes) in Todd 1996 (Example 1, p. 157 and Example 2b, p. 159) and Cohn 2000 (Examples 3 through 5, pp. 93-94).

[19] There are different definitions of parsimonious voice leading in the neo-Riemannian literature. Douthett and Steinbach 1998 provide a useful generalization of the concept.

realized as minor seconds, adhering to the principle that horizontal semitones are usually heard as different diatonic degrees.[20] The solfège patterns listed below each voice leading interpret the major or minor triad as a tonic.[21] Voice leadings in which moving voices ascend may be classified as *dominant*-type voice leadings (primary voice TI–DO), and those in which moving voices descend *subdominant*-type voice leadings (primary voice LE-SOL).[22] In textbooks, dominant-type voice leadings from augmented triads to major or minor triads are usually found in discussions of altered dominants, the augmented triad being interpreted as a major triad with chromatically raised 5th (V^{+}).[23] Subdominant-type voice leadings are not easily reconciled to a root-based perspective, since interpreting a subdominant-functioned augmented triad with a Roman numeral would depend upon a chromatically lowered root (in which case the analogous symbol might be "IV_"). Using root-neutral "*D*" and "*S*" designations obviates this difficulty.[24]

Representing these voice leadings on the clockface reveals pairwise relationships between voice leadings to major triads and

[20] The one exception is MI–ME, which I use in order to accommodate a solfège interpretation; most of Weitzmann's examples represent all moving voices as minor seconds. Cohn 2004 follows suit in his representations of voice leadings between hexatonic poles and mentions the principle on p. 306. In most conceptions of the chromatic scale as an embellished diatonic scale, there is no such object as $\downarrow\hat{4}$ (at least one student always cheekily suggests using "FE"). The interpretative problem here may be why of the four voice leadings, the one in which two voices descend is the rarest—indeed, I know of no examples in the literature. Interestingly, Louis and Thuille (1908, 319) exclude the three voice leadings in which 2 voices descend, perhaps an implicit acknowledgment of its musical rarity.

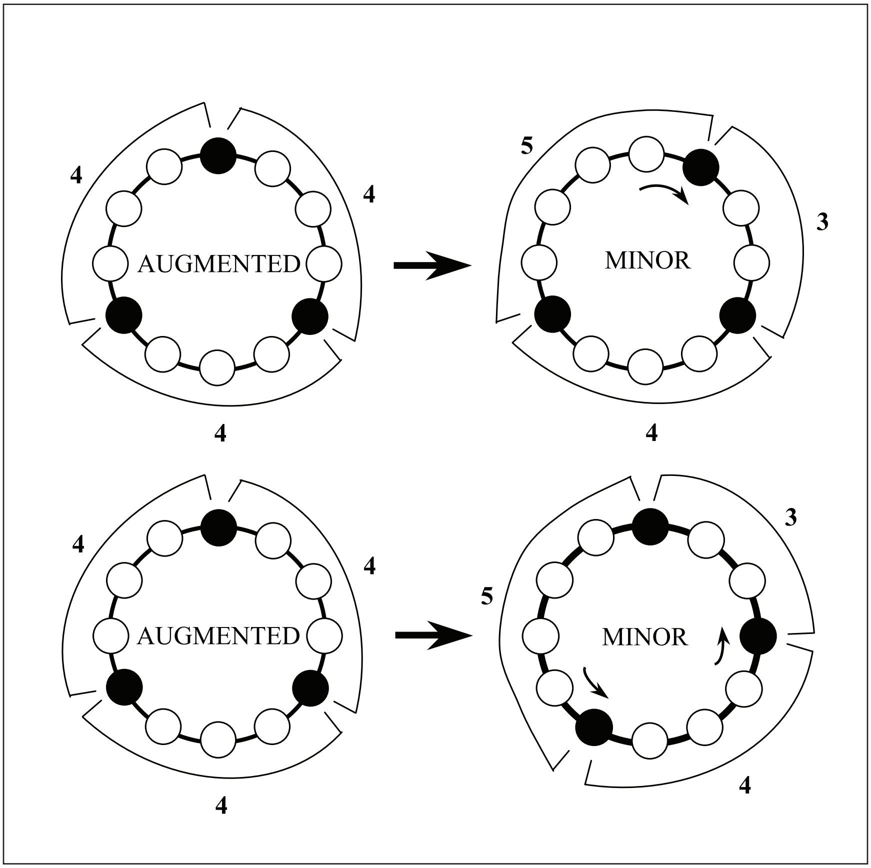
[21] Louis and Thuille (1908, 319) interpret the major triad reached by one descending voice as V in minor mode; the mixed scale degree in this case is the leading tone (in minor mode) instead of the lowered submediant (in major mode). Weitzmann (1853, 17) implicitly supports a tonic interpretation of the major triad in this voice leading by comparing the voice leading with that of vii^{o7} to I.

[22] As Cohn (2000, 92) explains, these functional relationships are inherent in Weitzmann's conception and in the dualist theoretic tradition in general.

[23] This interpretation runs into difficulty if the augmented triad resolves to a minor triad, however. Either the augmented triad must be spelled with $\sharp\hat{2}$ (as part of V^{+}) respelled as $\hat{3}$ (as part of i), or the augmented triad must be interpreted as III^{+}, a symbol and interpretation that has become outmoded.

[24] See Harrison 1994 for a recent revival of these functional labels.

to minor triads, respectively. As shown in Example 6, there are two ways to lead parsimoniously from an augmented triad (with interval pattern 444) to a *minor* triad (interval pattern 345)—by moving one voice upward or two voices downward. Similarly, there are two

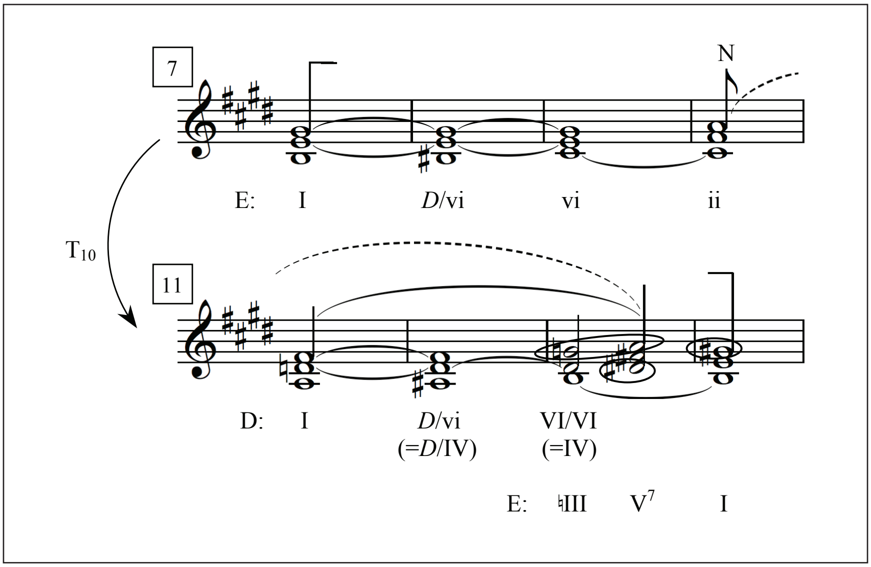


Example 6: Two ways to voice lead from an augmented triad (444) to a minor triad (345)

ways to lead parsimoniously from an augmented triad to a *major* triad—by moving two voices upward or one voice downward.

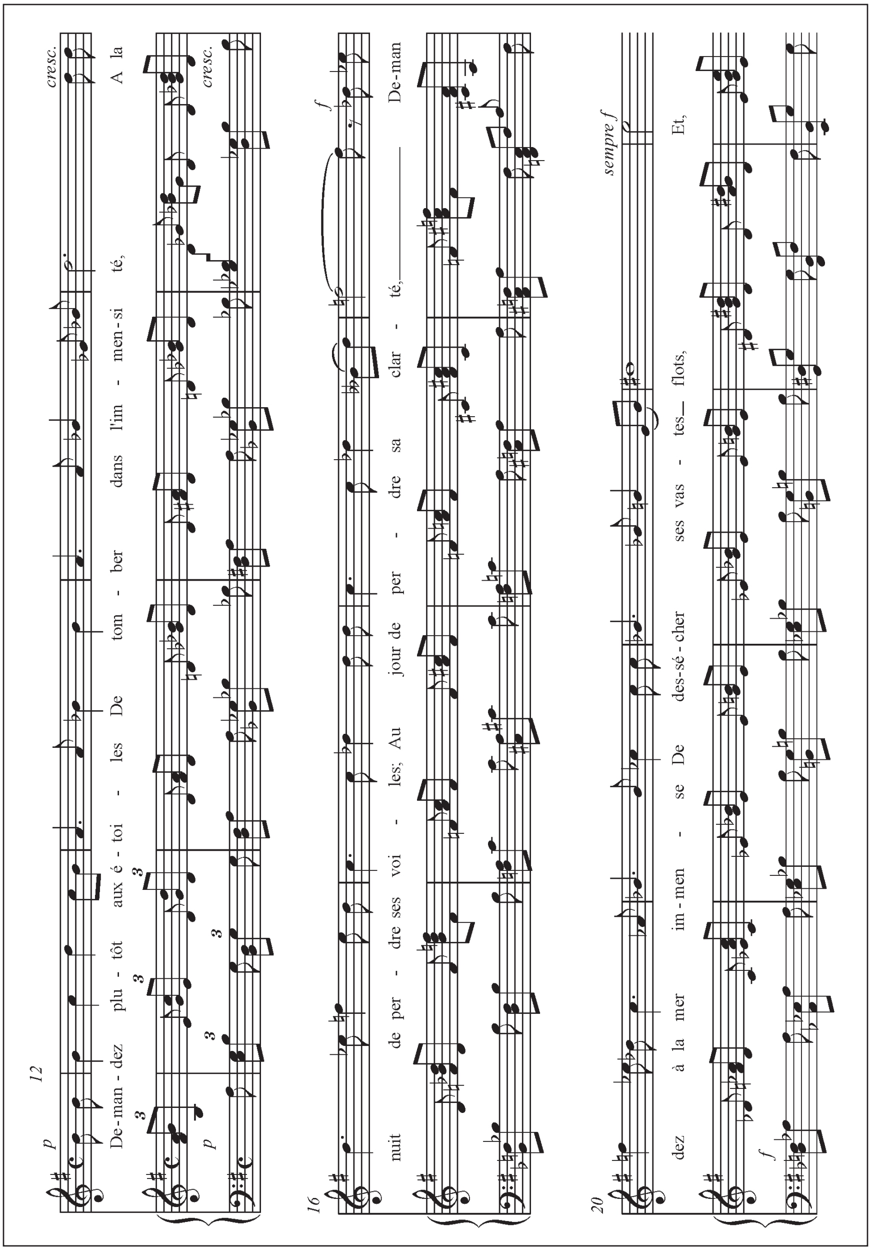
Two excerpts that feature enharmonic modulation using the augmented triad will serve to illustrate how the framework given in Example 5 proves useful. Each excerpt consists of a sequence whose successive patterns undergo alterations. A score and reduction of the first excerpt, from Liszt's *Sonetto 104 del Petrarca*, are given in Examples 7a and 7b, respectively. Although the reduction omits the bass line, the harmonic analysis reflects the actual bass.

Example 7a: Score of Liszt, "Sonetto 104 del Petrarca," mm. 7-14

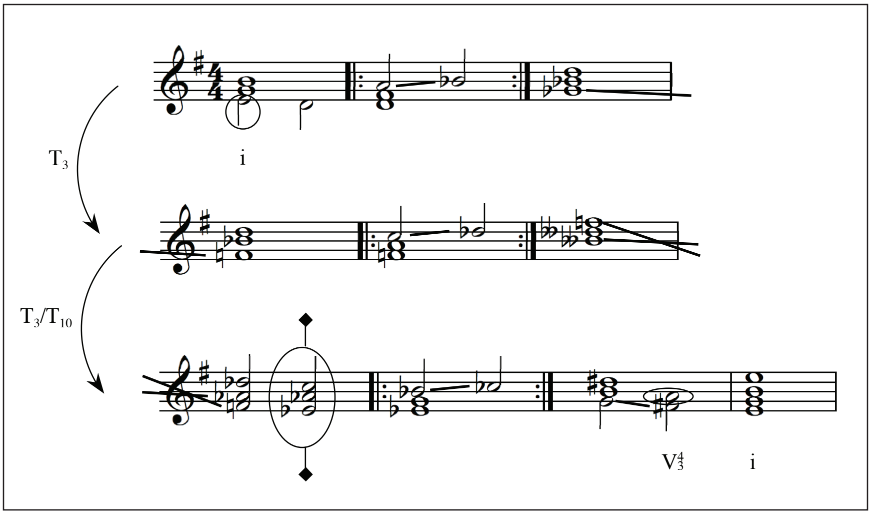


Example 7b: Reduction of Liszt, "Sonetto 104 del Petrarca," mm. 7-14

This excerpt consists of two four-measure subphrases that are transpositionally related; alterations in the second subphrase are circled on Example 7b. A dominant-functioned augmented triad appears in the second measure of each subphrase. In the first subphrase, only the bottom voice ascends (B♯–C♯); in the second subphrase, the bottom voice ascends as expected (A♯–B, respelled on the reduction to reflect the structural voice leading), but the upper voice unexpectedly ascends as well (F♯–G), producing a G-major triad on the downbeat. Whereas the first subphrase's augmented triad functions as *D* of vi, the second subphrase's augmented triad resolves deceptively to VI of vi, thereby functioning locally as *D* of IV. (The B♭ in the score conforms to the bass line, which prefigures IV through the descending arpeggiation D–B♭–G.) The other alterations to this measure include D♯ and A, both of which enable the heretofore-real sequence to return to tonic. Contrapuntally, the G♮ is a passing tone between F♯ (m. 11) and A (m. 13, b. 3), the latter of which is an upper neighbor to the structural G♯ that opens and closes the phrase. Though the melodic G♮ is contrapuntally subsidiary, its arrival corresponds to the most marked harmonic event of the passage, where IV in D major is reinterpreted as modally mixed ♮III in E major.



Example 8a: Score of Fauré, "Toujours," mm. 12-24



Example 8b: Reduction of Fauré, "Toujours," mm. 12-24

A score and reduction of the second excerpt, from Fauré's song "Toujours," op. 21, no. 2, are given in Examples 8a and 8b.[25] This excerpt exclusively employs the less common *S*-type voice leading.

In the poem, the narrator complains to his beloved, who has asked that he go away and forget about her. He indicates that he cannot possibly fulfill her request by comparing it to various unreasonable requests of nature: asking the stars to fall from the sky, the night to lose its darkness, the day to lose its light, and the sea to dry up.[26] The excerpt sets the first three requests; the enharmonic modulations come between each request, thereby linking each imagined upheaval in the natural world with tonal disorientation. The narrator's increasing agitation as he proceeds down the list is reflected in the sequence's ascending interval of transposition (up by minor third). As in the reduction of the Liszt excerpt, the bass line is omitted and circled pitches indicate alterations relative to the model. The second subphrase serves as the model for the sequence,

[25] This passage is singled out by Charles Burkhart as worthy of investigation in the 5th edition of his anthology (1994, 390); the piece does not appear in earlier or later editions.

[26] "Demandez plutôt aux étoiles / De tomber dans l'immensité, / A la nuit de perdre ses voiles, / Au jour de perdre sa clarté! / Demandez à la mer immense / De dessécher ses vastes flots."

with alterations occurring in the first and third subphrase.[27] In order to reveal the enharmonic modulations, the reduction spells moving voices with two different letter names and non-moving voices with the same letter names, following the same process used to construct Example 5.[28] In the first enharmonic reinterpretation, F♯ is reinterpreted as G♭; thus, *S* of D major is reinterpreted as *S* of B♭ major. The second enharmonic reinterpretation involves respelling A as B𝄫 to transform *S* of F major into *S* of D♭ major. The third pattern of the sequence inserts an A♭-major triad in its first measure, changing the interval of transposition beginning with that chord to T_{10} and enabling the phrase to cadence in the tonic.[29] The third enharmonic reinterpretation involves an intermediate

[27] It may seem counterintuitive to treat any pattern other than the first one as the model; a model in the sense used here is the pattern that contains no unique features. In employing the second pattern as the model of a three-pattern sequence, this sequence matches the opening of the Tristan Prelude (see Bass 1996, 282). The adjustments to the "Toujours" sequence, like those in the Tristan Prelude, similarly aid in reconciling a real sequence to a diatonic context.

[28] I do not mean to suggest that the spellings in the score are somehow wrong or invalid; it is often more expedient to spell an enharmonically reinterpreted chord only once in a score, which is what makes enharmonic modulation especially difficult for students. Recognizing this difficulty, Clendinning and Marvin (2005, 604) advise the student in a sidebar that "An enharmonic pivot chord can be spelled as it functions in the first key or as it functions in the second key. It may also appear twice, spelled once each way." The first and second augmented triads are spelled only one way: the spelling of the first reflects its *second* function; the spelling of the second reflects neither of its functions. That the latter is spelled in root position suggests a different kind of expediency: root-position augmented triads are easier to read (here I refer only to the right-hand voicings in mm. 17-18). The third augmented triad is spelled two ways, the first of which follows the expedient spelling of the second augmented triad, and the second of which—like the first augmented triad—reflects its *second* function. These spellings may not be Fauré's: the manuscript is in F♯ minor, and the two editions I have been able to examine (those in E minor and F minor) are inconsistent in their spellings of the augmented triads.

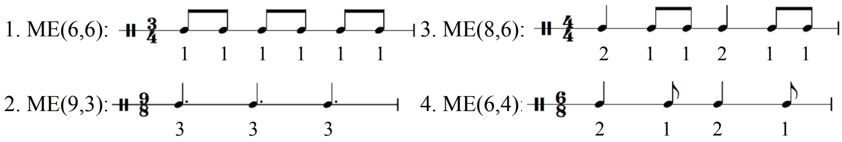
[29] It is a commonplace of Fauré's style to travel to distant tonal regions within a phrase, only to suddenly return to tonic at the very end. The adjustment of chord position on the D♭-major triad—root on top instead of chordal 3rd, as in the first two patterns—allows for smooth voice leading into the E♭-major triad with chordal 5th on top.

stage not shown in the reduction: G is respelled as A𝄫 in order to reinterpret *S* of E♭ major as *S* of C♭ major. The indicated change from flats to sharps allows the music to return notationally to its starting point.[30]

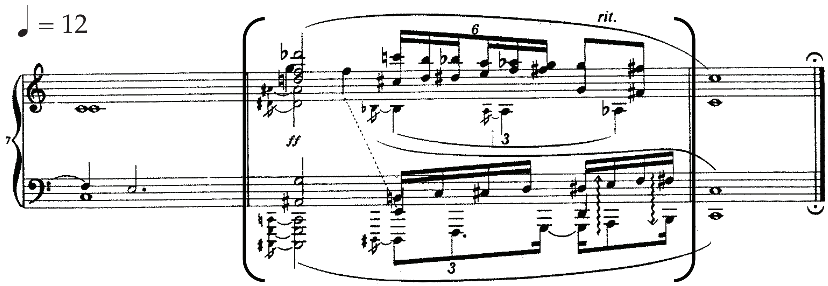
Maximal evenness can also usefully be studied in connection with rhythm, which naturally opens up the possibility of various other universes. Set members are durations, and the universe consists of the number of timepoints in a (notated or conceptual) measure. A shortest allowable duration must be chosen, analogous to the smallest interval in the pitch domain. Considering ME duration sets prompts the use of integers for durations, which is unfamiliar to even advanced students, who likely have not considered such a model since they first learned durations. (Here I am thinking of fundamentals exercises that ask, for example, how many sixteenth notes are in a doubly-dotted half note.) Various Class A and B ME duration sets are given in Example 9a; the eighth note is assigned a duration of length 1.

[30] See note 16.

a. Various Class A and B duration sets (♪ =1)



b. Charles Ives, "Bad Resolutions and Good Wan," mm. 7-9:
ME(12,2) + ME(12,3) + ME(12,4) + ME(12,6) + ME(24,3)



c. ME (8,3): first half of "son clave" pattern

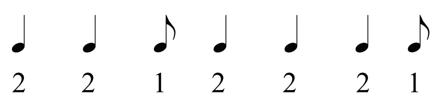
timepoints: 036

interval pattern: 332 or 

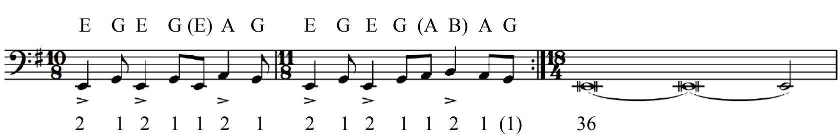
# of transpositions: 8

# of modes: 3

d. ME(12,7): Agbekor



e. ME(10,7) and ME (11,8): Naked City, "You Will Be Shot"



Example 9: Some ME rhythms

Sets 1 and 2 belong to Class A: they consist of equal divisions of measures that are six and nine eighth notes long, respectively. Sets 3 and 4 belong to Class B: each consists of a repeated pattern made up of two different durations. Example 9b provides the last three measures of Charles Ives' "Bad Resolutions and Good Wan" (1907), which superimposes several Class A sets: beats 3 and 4 of m. 8 collectively contain four equal divisions of a quarter-note beat and one equal division of a half note. More interesting are Class C duration sets, some examples of which are given in Examples 9c through 9e. Such duration sets are especially common in non-Western musics as well as Western popular music and jazz, providing an opportunity to draw on music outside the canon. A relatively common Class C rhythm is ME(8,3), which appears in Example 9c. The transposition and mode given in the example correspond to the first half of a "son clave" pattern, and is also the rhythm that students often produce when first attempting to perform 2-against-3 patterns.[31] As Traut 2005 shows, all three of its modes occur as accent patterns in 1980s popular music. Example 9d illustrates ME duration set (12,7), which is found in Agbekor, a music of the Ewe people of Ghana and Togo.[32] Finally, Example 9e is a transcription of an ostinato from Naked City's "You Will Be Shot," a piece that juxtaposes heavy metal and country genres.[33] The ostinato dominates the heavy metal sections, which frame the piece; the country sections are exclusively in (squarish) common time. The bar lines in my transcription reflect the two-fold repetition of pitch-class segment <EGEGAG> and duration segment <2121121>. The differences between the first two measures are indicated by the letter names and durations in parentheses: the pitch material of measure 2 modifies that of measure 1 in replacing the third E with

[31] The first half of a clave rhythm is usually represented by a quarter note, eighth rest, eighth note, quarter rest, quarter note in common time. Students produce this rhythm in a 2-vs.-3 context by trying to fit a threefold division into simple meter—8 is the first power of 2 that can be split into three integral parts that are roughly equivalent (3, 3, and 2).

[32] See Locke 2002 for an exposition of this tradition; this duration set is played by the gankogui, a type of bell. Hall (2005, 160-65) includes examples based upon ME(12,7) and other rhythms from the West African tradition. See Pressing 1983 and Rahn 1987 for other examples of duration sets that are isomorphic to familiar pitch sets; Cohn 1998, [7]-[8] also mentions these connections, referring to Pressing 1983.

[33] *Naked City*, Elektra/Nonesuch 979238-2, 1989. Naked City was a band led by John Zorn, active from 1988 to 1993.

A and B; the duration pattern of measure 2 adds an eighth note relative to measure 1. The integer model for rhythm that can be smoothly introduced by way of ME sets proves useful for studying rhythmic procedures in the music of Messiaen (see Hook 1998) and of Steve Reich, as both Cohn (1992) and John Roeder (2003) have shown.

III. Integrating Maximal Evenness with Other Topics: Combinatorics

The course continues by examining particular ME sets in detail, linking each set to musical excerpts and procedures: analytical forays are interspersed with continuing theoretical investigations.[34] In the continuing theoretical investigations, transpositional combination (henceforth TC) acts as a bridge to the complete roster of set classes.[35] A brief detour into combinatorics sets the stage; although knowledge of combinatorics is not crucial to an understanding of TC, I introduce the topic formally at this point because of its relevance to later concepts.[36] Having already manually counted transpositions and modes of ME sets and employed formulas for doing so, students are ready and open to the subject by this point in the course.

There are two things to keep in mind in counting objects selected from a (usually larger) set: whether the objects are *ordered* or *unordered*, and whether or not a given object can appear more than once in a set. Example 10 follows the usual order of introduction of the four categories, listing their associated formulas along with problems that we work through in class.

The boxes on the right side of the example are not part of the handout proper, but indicate some relevant applications of the concepts to which I will return.

[34] For a recent approach to the analysis of post-tonal music, see Alegant and Sly 2004.

[35] Transpositional combination is introduced in Cohn 1987.

[36] Enumeration is a long-standing tradition in music theory. See Hook 2007 for historical precedents and a thorough primer on combinatorics in music theory.

PERMUTATIONS and COMBINATIONS

$\downarrow$ **ordered** objects

$\downarrow$ **unordered** objects

n = the set of objects from which we're selecting

r = the number of objects we're selecting from n objects

$n! = n \times (n-1) \times (n-2) \ldots \times 1$ (n! = "n factorial"; 4! = 24; 0! = 1, by def.)

- PERMUTATIONS: $\dfrac{n!}{(n-r)!}$; if $n = r$, then $n!$ — twelve-tone rows
  - Example: n = 4, r = 4
  - Example: n = 4, r = 2
- COMBINATIONS: $\dfrac{n!}{(n-r)!\,r!}$ "n choose r" or $\dbinom{n}{r}$ — transpositional combinations; ic vector; subsets of a given cardinality
  - Example: n = 4, r = 4
  - Example: n = 4, r = 2
  - The number of double ice cream cones with **2 different** flavors, 31 total flavors
- PERMUTATIONS **WITH REPETITION**: n^r — pitch-class sets in a universe
  - Example: the number of ways to light this classroom
  - Example: the number of double ice cream cones, 31 flavors (vanilla + chocolate ≠ chocolate + vanilla)
- COMBINATIONS **WITH REPETITION**: $\dfrac{(r+n-1)!}{(n-1)!\,r!}$ or $\dbinom{r+n-1}{n-1}$ — harmonic sequences
  - Example: the number of double ice cream cones, 31 flavors (vanilla + chocolate = chocolate + vanilla)
  - Example: the number of lighting *levels* in this classroom

Example 10: Handout on permutations and combinations

I first spend some time in class deriving the first two formulas. Again using students, I ask a volunteer—let's call him John—to stand at the front of the room. How many different ways can John form a line? Obviously only one way. I then ask a second volunteer—let's call her Mary—to join John. How many different ways can John and Mary form a line? There are two possibilities: either John can stand in front or Mary can stand in front. Thus, for two objects, there are two possible orderings. Then a third volunteer—let's call her Robin—joins John and Mary. The situation grows slightly more complicated at this point: with John at the front of the line, there are two possibilities for the rest of the line (Mary, then Robin or Robin, then Mary); with Mary at the front of the line, there are again two possibilities; and with Robin at the front of the line there are two possibilities. Thus, for three objects there are six possible orderings. It gradually becomes clear that the number of orderings of n objects is the product of n times $n - 1$ times $n - 2$ and so on down to 1. This product is given by $n!$ (n factorial). If one wishes to know the number of orderings of a certain number of objects *selected from a larger number of objects*, then one multiplies only the number of products that one is selecting. For example, in selecting three objects from five, the number of permutations is 5 times 4 times 3, or n times $n - 1$ times $n - 2$. A shorthand for this is n times $n - 1$ and so on until $n - r + 1$, which may be abbreviated using factorial as the first formula given in the handout.

The only difference in the second formula—that for combinations—is the $r!$ in the denominator, which eliminates the different permutations of the r objects selected. The case of $n = 4$ and $r = 2$ will serve to illustrate. There are $\frac{4!}{(4-2)!}$ (= 12) permutations of 2 objects selected from 4 objects, but $\frac{4!}{(4-2)!2!}$ (= 6) combinations of 2 objects selected from 4 objects.[37]

I then give an assignment reproduced as Example 11. This assignment integrates students' knowledge of ME pitch sets in the chromatic universe with their new understanding of counting theory.

[37] For other derivations of these formulas, see, for example, Bose and Manvel 1984, Chapter 1 and Hook 2007.

a. assignment

1. Determine which sets can be produced by combining any two transpositions of ME(12,2).
2. Determine which sets can be produced by combining any two transpositions of ME(12,3).
3. Determine which sets can be produced by combining any two transpositions of ME(12,4).

Play all the resulting sets (as chords) on a keyboard and (as melodies) on your instrument.

Be sure to try all possible combinations! Some of the different combinations will produce identical results; prune these duplicates. In other words, if you happen upon a transposition of a set you've already produced, do not repeat it.

List the new sets in terms of their pc numbers, then enumerate these properties of each new set:

- How many distinct transpositions does it have?
- What is its interval pattern?
- How many modes does it have?
- Does it have a familiar name?
- If you produce another ME set, include it as well, identifying it as such. You need not list its properties, since we all know them now. (This means that there are some relationships between ME sets of different cardinalities other than complementation!)

b. answer to assignment (1. only)

Combining two transpositions of ME(12,2), the tritone:

There are six transpositions of ME(12,2): 06, 17, 28, 39, 4T, and 5E.

There are fifteen ways to combine six objects two at a time: $\binom{6}{2} = 15$

Combining two tritones that differ by 1 produces a transposition of [0167]: {0167}, {1278}, {2389}, {349T}, {45TE}, {56E0}.

Combining two tritones that differ by 2 produces a transposition of [0268]: {0268}, {1379}, {248T}, {359E}, {46T1}, {5702}. Familiar names: French augmented-sixth chord, dominant-7th chord with lowered 5th.

Combining two tritones that differ by 3 produces a transposition of [0369], i.e., ME(12,4): {0369}, {147T}, {258E}.

Example 11: Assignment on transpositional combination of (some) ME sets

The relevant formula is the one for combinations, which is known informally as "*n* choose *r*." Combining all possible pairs of transpositions of Class A ME sets uncovers new sets of musical interest whose properties can then be examined, and also reveals interconnections between certain ME sets of different cardinalities. As shown in the bottom half of the example, ME(12,2) at T_1 produces members of set class [0167], a set class featured in music of Bartók and Webern, among others; ME(12,2) at T_2 produces members of [0268], the prime form of the French augmented-sixth chord or the dominant-7th chord with lowered 5th; and ME(12,2) at T_3 produces ME(12,4). Combinations of ME(12,3) include ME(12,6) and hexatonic.[38] Combinations of ME(12,4) produce only ME(12,8).

Transpositional combination also dovetails nicely with the concept of transpositional symmetry. All ME sets in Class A and B are transpositionally symmetric (as can be seen in the sample interval patterns given in Example 2), as are some TC sets with ME constituents, including all the sets produced in the above assignment.[39]

In addition to its value for TC exercises, "*n* choose *r*" also proves useful in connection with creating interval-class vectors. In this case, *n* equals the cardinality of the pcset and *r* equals 2. The formula serves as a helpful "check step," ensuring that the interval-class vector contains the correct number of ics for that set cardinality. That *r* can take on any value from 1 to *n* reinforces the concept than an interval class is a subset of cardinality 2. While it is relatively easy to count subsets of cardinality 2 without the aid of the formula, counting subsets of larger cardinalities is somewhat more difficult. In listing all the trichordal pcsets of a particular hexachordal pcset, it is helpful to know that there are "6 choose 3" (= 20) trichordal pcsets in any hexachordal pcset.

A useful heuristic that summarizes information about subset counts is Pascal's Triangle, a modified form of which is given in Example 12.

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[38] See Cohn 1996 and 2004 for a generous sampling of excerpts that employ the hexatonic collection.

[39] Transpositionally symmetric sets can be produced by transpositional combinations of one or more complete interval cycles: see Cohn 1991. TC sets that are not constructed from complete interval cycles are not transpositionally symmetric. For example, pcset {0347}, which can be understood as 2 interval-class 3s at T_4 or 2 interval-class 4s at T_3, is not transpositionally symmetric because neither a single ic 3 nor a single ic 4 constitutes a complete interval cycle. The converse and more difficult exercise of disassembling larger sets into possible TC constituents—as given, for example, in Straus (2005, 117)—is made easier by the introduction of TC early in the course.

How many subsets of various sizes are contained within a set of a certain size?

As you know, the number of combinations of n things taken r at a time equals $\frac{n!}{(n-r)!r!}$.

n is the cardinality of the set, and r is the cardinality of the subset.

The total number of subsets of a set of cardinality n is 2^n (since a pitch class can either be present or absent)—the number of pitch-class sets of cardinality 0 through n.

cardinality of subset (r)

cardinality of set (n)	0	1	2	3	4	5	6	7	8	9	10	11	12	total (2^n)
1	1	1												2
2	1	2	1											4
3	1	3	3	1										8
4	1	4	6	4	1									16
5	1	5	10	10	5	1								32
6	1	6	15	20	15	6	1							64
7	1	7	21	35	35	21	7	1						128
8	1	8	28	56	70	56	28	8	1					256
9	1	9	36	84	126	126	84	36	9	1				512
10	1	10	45	120	210	252	210	120	45	10	1			1024
11	1	11	55	165	330	462	462	330	165	55	11	1		2048
12	1	12	66	220	495	792	924	792	495	220	66	12	1	4096

<u>Problem:</u>

List all trichordal pcsets in pcset {014589}. To what set class does each pcset belong? How many representatives of each SC are there?

<u>Solution:</u>

How many trichordal pcsets are contained within a hexachord?

$$\frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6\times 5\times 4}{3\times 2} = 20$$ (shaded value in table above)

Pcset {014589} contains {014}+{589}; {015}+{489}; {018}+{459}; {019}+{458}.
{045}+{189}; {048}+{159}; {049}+{158}.
{pc sets} {058}+{149}; {059}+{148}.
{089}+{145}.

[014][014]; [015][015]; [015][015]; [014][014].
{set classes} [015][015]; [048][048]; [037][037].
[037][037]; [037][037].
[014][014].

Any pcset in set class [014589] thus contains 6 pcsets in set class [014], 6 pcsets in set class [015], 6 pcsets in set class [037], and 2 pcsets in set class [048].

Example 12: Handout on subsets

Each number is the sum of the one above it and the one above and to its left, making it easy to construct.[40] Summing the numbers in each row gives the total number of subsets of a given cardinality. For instance, the total number of subsets of a set of cardinality 4 equals 16 (= 1 + 4 + 6 + 4 + 1). That the total number is always a power of 2 is not a coincidence, and a useful analogy can be made to light switches in a room. The problem may be framed as follows. Assume that there are three sets of lights, each controlled by a different switch. How many different ways to light the room are there? Each set of lights may be on or off, and the switches are independent of one another. In this case, the set of objects n is {ON, OFF}, from which three selections are made (three light switches). Since each switch is independent, there are 2x2x2 = 8 possible ways to light the room. Returning to Example 10, the formula for the number of permutations *with repetition* is n^r. Counting "1" as ON and "0" as OFF, the 8 possibilities are given in Example 13.

Complementary Pairs	Binary Representation	Number of Lights On
Pairs	Representation	Lights On
	000	0
	001	
	010	1
	100	
	011	
	101	2
	110	
	111	3

Example 13: Light-switch analogy for pcsets

[40] The vertical columns in the table correspond to the left diagonals of the triangle as it is usually represented. In the triangle, each interior number is the sum of the two numbers above and to its left and right. This is given by a well-known binomial identity; see Tucker 2002, 214.

By extension, on the clockface a pitch class is either present or absent, two states that are modeled by filled and unfilled circles, respectively; there are thus $2^{12} = 4096$ pitch-class sets.[41]

Such a broad perspective can be helpful in teaching the concept of the set class, a concept with which many students struggle. That there are far fewer set classes than pitch-class sets goes a long way towards selling the concept. A very rough approximation of the number of set classes can be obtained by dividing 4096 by 24 (= 170 2/3), the maximum number of pcsets in a set class (12 transposed forms plus 12 inverted forms); the result is less than the actual number of set classes (224) because a small number of set classes have non-trivial transpositional symmetry (16) and a much larger number of set classes possess inversional symmetry (96).[42] Of the eight non-trivial ME sets in the chromatic universe, six have transpositional symmetry; three other transpositionally symmetric sets are familiar from the TC assignment. Thus, nine of the sixteen transpositionally symmetric sets have already been introduced.[43] Inversional symmetry, on the other hand, is unfamiliar—at least in pc space—but ME sets can serve as test cases since they are inversionally symmetrical by definition.[44] The concept of the set class has thus been lurking since the initial investigation of maximal evenness, in which each ME(12,*n*) set-class was referred to as a single object.

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[41] This view of pitch-class sets is often the basis for programming routines that take pitch-class sets as their objects. See Brinkman 1990, 625-28 for a discussion. That the sum of all subsets is a power of 2 is given by another binomial identity. See Tucker 2002, 216-17 and Hook 2007, [30]-[31].

[42] All sets have trivial transpositional symmetry: T_0 preserves any set. Calculating the number of set classes from the cardinality of the universe, the cardinality of pcset, and the nature of the canonical operations is fairly complicated and therefore properly lies outside the scope of the class. See Hook 2007 for an explanation of the underlying mathematics and Nolan 2003 for a fascinating survey of historical precedents.

[43] The 7 others include [012678] and [013679]; the complements of [06], [0167] and [0268]; and the null set and aggregate, which are trivial.

[44] Clough and Douthett give a proof in Theorem 1.8 (pp. 109-10).

In teaching inversional symmetry, it is helpful to imagine that the clockface is a necklace. Indeed, a well-known counting problem in mathematics involves counting the number of different necklaces containing a certain number of beads that come in a certain number of colors. The number of set classes (under transposition and inversion) is equivalent to the number of necklaces of 12 beads that come in two colors—the two colors corresponding to the presence or absence of a pitch class, respectively. Transposition corresponds to rotating the necklace, while inversion corresponds to flipping the necklace over.[45]

From the pictures of ME sets in the 12-element universe (Example 1b), it is apparent that Class A and Class B ME sets are inversionally symmetrical. The inversional symmetry of non-trivial Class C ME sets is less obvious, so carefully demonstrating the inversional symmetry of ME(12,5) and ME(12,7) by trying out different inversional axes can prove beneficial. The same can be done for all non-trivial ME sets in the 7-element universe, which all belong to Class C.

Another common way to demonstrate inversional symmetry is by examining the interval pattern: a palindromic interval pattern represents an inversionally symmetric set.[46] For students who have trouble visualizing axes on the clockface, this method is useful and often quicker, though it has two drawbacks. First, set classes fall into three different categories with respect to the way in which the palindromic interval pattern may be represented:

[45] Such a problem is a subset of those involving counting the number of colorings of faces, sides or corners of various geometric figures; solving such problems involves determining all the rotational and reflective symmetries of the figure being colored. See Tucker 2002, Chapter 9 and Hook 2007.

[46] See, for example, Straus 2005, 87. This method is also used to determine if two pcsets are inversionally related.

1. The palindrome necessarily involves *all* intervals in the pattern. Example: [027], with interval pattern <525>. This category includes 12 of the 80 set classes of cardinality 3 through 9 (15%).

2. The palindrome necessarily involves *all but one* of the intervals in the pattern. Example: [0268], with interval pattern <242> or <424>. For such sets, there are always two ways to represent the palindrome, both of which exclude one interval in the pattern. This category includes 36 of 80 set classes (45%).

3. The palindrome can either involve all intervals in the pattern or all but one interval. Example: [01348], with interval pattern <1441> or <41214>. This category includes 32 of 80 set classes (40%).

Thus, if the palindrome is not immediately apparent in the way the interval pattern is initially represented, it is necessary to consider the other possibility. Only one ME set in the 12-element universe falls into the second category above, and for that reason serves as a useful foil to the rest: the only representations of ME(12,8)'s interval pattern as a palindrome are 2121212 and 1212121, both of which exclude one interval. All other ME sets in the 12-element universe fall into the third category: for example, ME(12,9)'s palindromic interval pattern may be represented as 11211211 (excluding one interval) or 121121121 (including all intervals). Trying out the two possibilities for ME sets helps prepare students for encounters with sets that fall into the first two categories.

The second drawback to using the interval pattern to identify inversional symmetry is that the palindrome is not always apparent in the normal form of a pcset. Again, ME(12,5) and ME(12,7) serve as two examples: for ME(12,5), the normal-form interval pattern is 22323, but the palindrome is either 23232 (including all intervals) or 3223 (excluding one interval). ME(7,4) and ME(7,5) serve as further examples.

The two enumeration formulas not yet linked with a musical application involve counting permutations (*without* repetition) and combinations *with* repetition, the first and last formulas in Example 10. The last formula is least relevant in the course, and I do not spend time in class deriving it, but include it for the sake of completeness.[47] Permutations, on the other hand, are relevant for counting ordered sets of any cardinality—most particularly for twelve-tone rows. Spending time working out that there are 12! = 479,001,600 distinct twelve-tone rows gives students a perspective that can later be modified upon the introduction of twelve-tone operations and the equivalence classes that result. It turns out that an approximation of row classes, 479,001,600 / 48 (= 9,979,200), is much closer to the actual number of row classes (9,985,920) than the corresponding approximation of set classes.[48] That there are far more twelve-tone rows than set classes drives home the point that all unordered sets can be ordered in multiple ways. This perspective leads to fruitful discussions about what constitutes an "interesting" row class—there are, after all, many "boring" rows—and how to construct row classes with compositionally useful properties.[49]

As the full apparatus of pitch-class set theory is erected, ME sets continue to serve as landmarks. There is one (and only one) ME set in each column of the set-class table, and with the exception of ME(12,5) and (12,7), ME sets appear at the bottom of the column. Sets of minimal evenness—chromatic clusters—appear at the top of each column. Other sets can be related to ME sets in various ways. Set class [0148], used frequently in the music of Schoenberg, is easily viewed as ME(12,3) plus an "extra" pc that lies adjacent to one of the pcs of ME(12,3). Scriabin's "mystic chord," a member of set class [013579], can be conceptualized as a "minimally perturbed" version of the whole-tone scale: a whole-tone scale with one pc nudged up or down a half step.[50]

[47] There are musical contexts in which multisets are relevant: see Lewin 1998 (62-67) on pcsets with doublings; see Ricci 2002 and 2004 for a discussion of harmonic sequences conceived as concatenations of root motions.

[48] See Hook 2007 for a proof of the number of twelve-tone row classes.

[49] See Hunter and von Hippel 2003 for a study of the extreme rarity of symmetry in twelve-tone row classes in general.

[50] See Callender 1998. Another well-known minimal perturbation of an ME set is [0258] (major-minor and half-diminished 7th chord), a perturbation of ME(12,4); see Childs 1998. Such relationships have significant implications for voice leading, as demonstrated earlier in connection with enharmonic modulation using the augmented triad.

According to Bain, another essential component of the natural critical learning environment is to leave students with a question. Whether or not we explicitly seek an answer to the question within the framework of the course, it is certainly implicit throughout, providing strong motivation for our continuing investigations: Why *are* maximally even sets so prevalent, both in the pitch and rhythmic domains? Exploring and enumerating their properties and relating them to each other provides part of an answer. Of course, there is as yet no definitive answer to this question, and it is probable that no such answer exists. And letting students know that there are such open questions in the field of music theory seems a fitting way to conclude the music theory core sequence.

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